Pure projective tilting modules

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In memory of Gena Puninski, friend and colleague

Abstract. Let T_R be a 1-tilting module with tilting torsion pair $(\operatorname{Gen} T, \mathcal{F})$ in Mod-R. The following conditions are proved to be equivalent: (1) T is pure projective; (2) $\operatorname{Gen} T$ is a definable subcategory of Mod-R with enough pure projectives; (3) both classes $\operatorname{Gen} T$ and $\mathcal{F}$ are finitely axiomatizable; and (4) the heart of the corresponding HRS t-structure (in the derived category $\mathcal{D}^b(\text{Mod-}R)$) is Grothendieck. This article explores in this context the question raised by Saorín if the Grothendieck condition on the heart of an HRS t-structure implies that it is equivalent to a module category. This amounts to asking if T is tilting equivalent to a finitely presented module. This is resolved in the positive for a Krull-Schmidt ring, and for a commutative ring, a positive answer follows from a proof that every pure projective 1-tilting module is projective. However, a general criterion is found that yields a negative answer to Saorín's Question and this criterion is satisfied by the universal enveloping algebra of a semisimple Lie algebra, a left and right noetherian domain.

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Introduction

In their study [5] of perverse sheaves, Beilinson, Bernstein and Deligne introduced the notion of a t-structure on a triangulated category $\mathcal{D}$ and showed that the triangulated structure of $\mathcal{D}$ induces the structure of an abelian category on the heart $\mathcal{H}_t \subseteq \mathcal{D}$ of the t-structure. In the case when $\mathcal{D} = \mathcal{D}^b(\mathcal{G})$ is the bounded derived category of a Grothendieck category, their results [5, Proposition 3.1.10] provide an exact functor $\mathcal{D}^b(\mathcal{H}_t) \to \mathcal{D}^b(\mathcal{G})$, called the realization functor, that extends the inclusion $\mathcal{H}_t \subseteq \mathcal{D}^b(\mathcal{G})$. For a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$, Happel, Reiten, and Smalø [17] introduced a t-structure on the bounded derived category $\mathcal{D}^b(\mathcal{G})$ called the HRS t-structure on $\mathcal{D}^b(\mathcal{G})$, and they proved [17, Theorem 3.3] that if $(\mathcal{T}, \mathcal{F})$ is a torsion tilting pair, then the realization functor $\mathcal{D}^b(\mathcal{H}_t) \to \mathcal{D}^b(\mathcal{G})$ is an equivalence of triangulated categories. In their study of what properties are preserved by this kind of derived equivalence, Parra and Saorín ([27] and [28, Theorem 1.2]) showed that the heart $\mathcal{H}_t$ of the HRS t-structure on $\mathcal{D}^b(\mathcal{G})$ induced by a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$ is itself Grothendieck if and only if the torsion free class $\mathcal{F}$ is closed under direct limits in $\mathcal{G}$. In a module category Mod-R, tilting torsion pairs $(\operatorname{Gen} T, \mathcal{F})$ arise from a 1-tilting module T, which prompted Saorín to ask whether the realization functor is a derived Morita equivalence, in the following form.

Question 1. [27, Question 5.5] *Let R be a ring, T_R a 1-tilting R-module, and $\mathcal{H}_t \subseteq \mathcal{D}^b(\text{Mod-}R)$ the heart of the HRS t-structure induced in $\mathcal{D}^b(\text{Mod-}R)$ by the tilting torsion pair $(\operatorname{Gen} T, \mathcal{F})$. If $\mathcal{H}_t$ is Grothendieck, is it equivalent to the module category* Mod-S *for some ring S?*

The question of whether the heart of an HRS t-structure induced by a torsion pair in Mod-R (resp., mod-R) is equivalent to a module category has been explored by several authors [17, 6, 1]. To formulate Saorín's Question in terms of module theory, we show (Corollary 2.5) that if T is a 1-tilting module, then the heart $\mathcal{H}_t \subseteq \mathcal{D}(\text{Mod-}R)$ of the HRS t-structure induced by the tilting torsion pair $(\operatorname{Gen} T, \mathcal{F})$ in Mod-R is a Grothendieck category if and only if T is pure projective. Recall that a Grothendieck category is equivalent to a module category if and only if it contains a finitely generated projective generator. Corollary 2.7 implies that this is the case if and only if T is a *classical* 1-tilting module, defined to be a tilting module tilting equivalent to a finitely presented module. These results may also be seen as special cases of the work of Angeleri, Marks and Vitoria [1, Theorem 3.7 and Lemma 2.9], and they suggest the following module theoretic formulation of Saorín's Question, which we take up in this article.

Question 2. (Saorín, pure projective version) *For which rings R is every pure projective 1-tilting module classical?*

Saorín's Question comes about from considerations that arise in the derived Morita theory of rings, but the notion of a pure projective 1-tilting module is a natural one to study in any case, because it represents the situation dual to the result of the first author [2] (generalized by Šťovíček [36] to all cotilting modules), that every 1-cotilting module is pure injective. On the other hand, there are many examples of 1-tilting modules that are not pure projective.

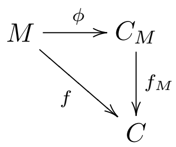
The pure projective version of Saorín's Question is related to the study [19] of definable subcategories with enough pure projective modules. It is well known that if $\mathcal{C} \subseteq \text{Mod-}R$ is a definable subcategory and $M \in \mathcal{C}$, then there exists a pure monomorphism $m : M \to U$ with U a pure injective module in $\mathcal{C}$. A definable subcategory $\mathcal{C} \subseteq \text{Mod-}R$ is said to have *enough pure projective modules* if for every module $M \in \mathcal{C}$, there exists a pure epimorphism $e : V \to M$ with V a pure projective module in $\mathcal{C}$. Such definable subcategories of Mod-R arise "classically" (cf. [8, 24]) as the categories $\mathcal{C}$ obtained by taking the closure under direct limits $\mathcal{C} = \varinjlim(\mathcal{X})$, of a covariantly finite subcategory $\mathcal{X} \subseteq \text{mod-}R$ of finitely presented modules. As part of our analysis of a pure projective 1-tilting module T, we show (Theorem 2.4) that the class $\mathcal{T} = \text{Gen}\, T = T^{\perp}$ of Mod-R, which is also definable [3], has enough pure projective modules, and (Theorem 2.6) that if T is a classical 1-tilting module, then $\mathcal{T} = \varinjlim(\mathcal{X})$, with $\mathcal{X} \subseteq \text{mod-}R$ covariantly finite.

The pure projective version of Saorín's Question has a positive answer for a large class of rings. By Corollary 2.8, if R is a ring over which every pure projective module is a direct sum of finitely presented modules, then every 1-tilting module T is classical. This includes the *Krull-Schmidt* rings, the rings over which every finitely presented module is a direct sum of modules with a local endomorphism ring. For a general commutative ring, we use a henselization argument (Theorem 3.7) to show that every pure projective 1-tilting module is projective. Considerably easier proofs are provided for a noetherian commutative ring or an arithmetic ring.

But the answer to Saorín's Question is not affirmative in general. In Theorem 4.3, we identify a criterion for an idempotent ideal I of R, finitely generated on the left, sufficient to yield a pure projective 1-tilting module T that is not classical. The construction is an elaboration of J. Whitehead's method of producing a projective module whose trace is such an idempotent ideal. The tilting class that arises is given by $\mathcal{T} = \{M \in \text{Mod-}R \mid M = MI\}$. For example, if $R = U(L)$ is the universal enveloping algebra of a semisimple Lie algebra L and I is the annihilator of the trivial module $\mathbb{C}_L$, then the conditions of Theorem 4.3 are met, so that one obtains a nonclassical pure projective 1-tilting module $T_{U(L)}$ over a domain that is both left and right noetherian.

A preliminary section of the article is devoted to the rôle that definable subcategories of Mod-R play in torsion theory. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair in Mod-R, then the torsion free class $\mathcal{F}$ always has direct limits, given by the torsion free quotient module of a direct limit in Mod-R. By the work of Parra and Saorín, the heart of the associated HRS t-structure is Grothendieck if and only if the direct limits in $\mathcal{F}$ coincide with those in the ambient category Mod-R, that is, if $\mathcal{F}$ is *closed under direct limits in* Mod-R. This is equivalent to the condition that the torsion free class $\mathcal{F}$ is a definable subcategory of Mod-R, or that the torsion radical of $(\mathcal{T}, \mathcal{F})$ respects direct limits. Hereditary torsion pairs with this property are called *elementary* [31, §11.1.3]. We call a torsion pair $(\mathcal{T}, \mathcal{F})$ for which both $\mathcal{T}$ and $\mathcal{F}$ are definable a *coherent* torsion pair, in view of Theorem 1.4, which implies that this condition is equivalent to the torsion radical being a coherent functor. If T is a pure projective 1-tilting module, then Corollary 2.5 and [3] imply that the tilting torsion pair $(\operatorname{Gen} T, \mathcal{F})$ is coherent.

In what follows, R is an associative ring with unit, Mod-R denotes the category of right R-modules, and mod-R the subcategory of finitely presented right R-modules. The category of abelian groups is denoted by Ab. For a subcategory $\mathcal{C}$ of Mod-R, $\operatorname{Add}(\mathcal{C})$ (resp., $\operatorname{add}(\mathcal{C})$) denotes the class of modules isomorphic to a summand of a (resp., finite) direct sum of modules in $\mathcal{C}$. If the class $\mathcal{C} = \{T\}$ is singleton, we write $\operatorname{Add} T$ (resp., $\operatorname{add} T$). Similarly, $\operatorname{Gen}(\mathcal{C})$ (resp., $\operatorname{Gen} T$) denotes the class of epimorphic images of direct sums of modules in $\mathcal{C}$ (resp., of copies of T).
Consider a subcategory $\mathcal{C}$ of Mod-R and a module $M \in \mathcal{C}$. A homomorphism $\phi : M \to C_M$ is a $\mathcal{C}$-*preenvelope* of M, if for every homomorphism $f : M \to C$ with $C \in \mathcal{C}$ there exists a homomorphism $f_M : C_M \to C$ such that the diagram



commutes. A $\mathcal{C}$-*preenvelope* $\phi : M \to C_M$ is a $\mathcal{C}$-*envelope* if every endomorphism $f : C_M \to C_M$ such that $f \circ \phi = \phi$ is an automorphism of C_M. We say that $\mathcal{C}$ is *preenveloping in* an additive category $\mathcal{A} \supseteq \mathcal{C}$ if every module $A \in \mathcal{A}$ has a $\mathcal{C}$-preenvelope. If $\mathcal{A} = $ Mod-R, we just say that $\mathcal{C}$ is *preenveloping;* if $\mathcal{A} = $ mod-R, we call $\mathcal{C}$ a *covariantly finite* subcategory of mod-R. The notions of a $\mathcal{C}$-*precover* and *precovering* are defined dually.
The superscript $\perp_i$ is used to denote orthogonality with respect to the bifunctor $\operatorname{Ext}^i_R(-,-)$, so if $C \in$ Mod-R, then $C^{\perp_i} = \operatorname{Ker} \operatorname{Ext}^i_R(C,-)$ and similarly for ${}^{\perp_i}C$; if $\mathcal{C} \subseteq$ Mod-R is a class, then $\mathcal{C}^{\perp_i} = \cap\{C^{\perp_i} \mid C \in \mathcal{C}\}$. The unadorned superscript $\perp$ refers to $\perp_1$ and $\perp_\infty$ will be used to denote the class orthogonal (on the appropriate side) with respect to all the bifunctors Ext^i_R, $i \geq 1$. A $\mathcal{C}$-preenvelope $\phi : M \to M_{\mathcal{C}}$ is *special* if it is a monomorphism with $\operatorname{Coker} \phi \in {}^{\perp}\mathcal{C}$; *special* $\mathcal{C}$-*precovers* are defined dually.
The language for right R-modules is $\mathcal{L}(R) = (+, -, 0, r)_{r \in R}$; the standard axioms for a right R-module are expressible in $\mathcal{L}(R)$ and the theory Th(Mod-R) of right R-modules consists of the consequences of these axioms. A subcategory $\mathcal{C} \subseteq$ Mod-R is *elementary* if it is the class of models of some collection $\Sigma \supseteq$ Th(Mod-R) of sentences in $\mathcal{L}(R)$, $\mathcal{C} = \operatorname{Mod}(\Sigma)$. Equivalently, the class $\mathcal{C} \subseteq$ Mod-R is *axiomatized* by Σ. An elementary class $\mathcal{E} \subseteq$ Mod-R is *finitely axiomatizable* (relative to the theory Th(Mod-R)) if there exists a sentence σ in $\mathcal{L}(R)$ such that $\mathcal{E} = $ Mod(Th(Mod-R) $\cup \{\sigma\}$) is the subcategory of Mod-R of modules M that satisfy σ, $M \models \sigma$.

1. Coherent torsion pairs

Definable subcategories of Mod-R were introduced by Crawley-Boevey [9] to characterize in non-logical terms the elementary additive classes of right R-modules, which arise in the model theory of modules and are in bijective correspondence with the closed subsets of the Ziegler spectrum [39].

Definition 1.1. A full subcategory $\mathcal{C} \subseteq$ Mod-R is *definable* if it is closed under products, pure submodules and direct limits.

Definable subcategories are also closed under direct sums, which may be regarded as direct limits of finite direct products, or pure submodules of direct products. There are several characterizations of definable categories that are useful. Recall that an additive functor F : Mod-$R \to$ Ab is *coherent* if it respects direct limits and direct products. Recall also that pure exact sequences arise as direct limits of split exact sequences, so that a coherent functor takes a pure exact sequence in Mod-R to one in Ab.

Proposition 1.2. *Let $\mathcal{C}$ be a full subcategory of* Mod-R. *The following are equivalent:*

(1) *$\mathcal{C}$ is definable;*
(2) *$\mathcal{C}$ is defined by the vanishing of some set of coherent functors;*
(3) *$\mathcal{C}$ is closed under direct products, pure submodules and pure epimorphic images; and*
(4) *$\mathcal{C}$ is an elementary class, closed under direct sums and direct summands.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (4) are from [9, §2.1, 2.3], where they are stated for an algebra over an infinite field; for the general case, one must include in Condition (4) that the elementary class is closed under direct sums.
(2) $\Rightarrow$ (3). This is a consequence of the fact that coherent functors are exact on pure short exact sequences (see [9, §2.1, Lemma 1]).
(3) $\Rightarrow$ (1). This holds because every direct limit is a pure epimorphic image of the direct sum of the modules in the directed system. □

An important consequence of Proposition 1.2(4) is that definable subcategories of Mod-R are closed under pure injective envelopes. This is because the pure injective envelope $M \to \mathrm{PE}(M)$ of a module M is an elementary embedding [31, Theorem 4.3.21], and therefore belongs to any elementary subcategory of Mod-R that contains M.
Suppose that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in Mod-R. The torsion free class $\mathcal{F}$ is already closed under products and submodules, so that it is definable if and only if it is closed under direct limits. On the other hand, the torsion class $\mathcal{T}$ is closed under coproducts and quotients, so that it is already closed under direct limits. It is therefore definable if and only if it is closed under pure submodules and products.

Definition 1.3. A torsion pair $(\mathcal{T}, \mathcal{F})$ in Mod-R is called *coherent* if the associated torsion radical t : Mod-$R \to$ Mod-R (composed with the forgetful functor to Ab) is coherent.

If the torsion radical t of $(\mathcal{T}, \mathcal{F})$ is a coherent functor, then both of the classes $\mathcal{T}$ and $\mathcal{F}$ are definable, because $\mathcal{F}$ is defined by the vanishing of t, while $\mathcal{T}$ is defined by the vanishing of the torsion free quotient functor, $M \mapsto M/t(M)$, which is also coherent. Indeed, by [9, Lemma 2, §2.1], every coherent subfunctor of the forgetful functor is given by a positive primitive formula $\tau(u)$ in the language $\mathcal{L}(R)$ of R-modules, so that $t(M) = \tau(M)$ for all R-modules M_R. This implies that the torsion class is axiomatized by the sentence $\forall u(\tau(u))$, so that $\mathcal{T} = \mathrm{Mod}\,(\forall u(\tau(u)))$, and that the torsion free class is axiomatized by the sentence $\forall u(\tau(u) \to (u \doteq 0))$. Thus both of the classes $\mathcal{T}$ and $\mathcal{F}$ are finitely axiomatizable.

Theorem 1.4. (cf. [30, Theorem 7]) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* Mod-R *with torsion radical t. The torsion free class $\mathcal{F}$ is definable if and only if t respects direct limits. The torsion class $\mathcal{T}$ is closed under direct products in* Mod-R *if and only if t respects products. Therefore, $(\mathcal{T}, \mathcal{F})$ is coherent if and only if both of the classes $\mathcal{T}$ and $\mathcal{F}$ are definable.*

Proof. If t respects direct limits, then it is clear that a direct limit of torsion free modules is torsion free. To prove the converse, suppose that a directed system M_i, $i \in I$, in Mod-R is given, with direct limit $M = \varinjlim M_i$. Each M_i is an extension of its torsion free quotient by its torsion submodule. These extensions themselves form a directed system of short exact sequences, with limit as shown in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t(M_i) & \longrightarrow & M_i & \longrightarrow & M_i/t(M_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varinjlim t(M_i) & \longrightarrow & M & \longrightarrow & \varinjlim M_i/t(M_i) & \longrightarrow & 0. \end{array}$$

A torsion class is closed under direct limits so that $\varinjlim t(M_i)$ is torsion, which induces a canonical morphism $\varinjlim t(M_i) \to t(\varinjlim M_i)$. By assumption, the direct limit $\varinjlim M_i/t(M_i)$ is torsion free, so that the short exact sequence in the bottom row represents M as the extension of its torsion free quotient by its torsion submodule, which implies that the canonical morphism is an isomorphism.

If t respects direct products, then it is clear that a direct product of torsion modules is torsion. To prove the converse, let N_j, $j \in J$, be a family of modules with product $N = \prod_j N_j$, and use a dual argument, but taking a product instead of a direct limit, to obtain the short exact sequence

$$0 \longrightarrow \prod_j t(N_j) \longrightarrow N \longrightarrow \prod_j N/t(N_j) \longrightarrow 0.$$

A torsion free class is closed under products, so the right term $\prod_j N_j/t(N_j)$ is torsion free, which induces a canonical morphism $t(\prod_j N_j) \to \prod_j t(N_j)$. By assumption, the product $\prod_j t(N_j)$ is torsion, so that the short exact sequence represents N as the extension of its torsion free quotient by its torsion submodule, which implies that the canonical morphism is an isomorphism.
If $\mathcal{T}$ and $\mathcal{F}$ are both definable, then these two statements imply that t respects direct limits and direct products. □

Corollary 1.5. *A torsion pair* $(\mathcal{T}, \mathcal{F})$ *in* Mod-R *is coherent if and only if both* $\mathcal{T}$ *and* $\mathcal{F}$ *are finitely axiomatizable.*

Proof. If $\mathcal{T}$ and $\mathcal{F}$ are both finitely axiomatizable, then they are elementary and additive, and therefore definable. □

Finitely axiomatizable definable subcategories correspond to basic Zariski open subsets of the Ziegler spectrum in the sense of [31, Chapter 14]. Examples of coherent torsion pairs will be found with the help of pure projective modules.

Definition 1.6. An R-module is *pure projective* if it has the projective property with respect to pure exact sequences. Other equivalent formulations are the following:

(1) A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented modules. For this reason, the subcategory of pure projective modules is denoted by Add(mod-R).
(2) A module M is pure projective if and only if every pure short exact sequence $0 \to A \to B \to M \to 0$ splits.

Corollary 1.7. *If* $(\mathcal{T}, \mathcal{F})$ *is a coherent torsion pair in* Mod-R, *then* $\mathcal{T} \cap {}^{\perp}\mathcal{T} \subseteq$ Add(mod-R).

Proof. Consider a pure short exact sequence $0 \to A \to B \xrightarrow{g} P \to 0$, with P in $\mathcal{T} \cap {}^{\perp}\mathcal{T}$ and B a direct sum of finitely presented modules. Apply the torsion radical,

$$0 \longrightarrow t(A) \longrightarrow t(B) \xrightarrow{t(g)} P \longrightarrow 0.$$

The sequence is exact, because t is a coherent functor, and the third term is $t(P) = P$. But $t(A) \in \mathcal{T}$ and $P \in {}^{\perp}\mathcal{T}$, so the sequence splits. If $s : P \to t(B)$ is a section of $t(g)$, then, because t is a subfunctor of the identity functor on Mod-R, the composition $s : P \to t(B) \subseteq B$ is a section of $g : B \to P$. □

Recall from the Introduction that a definable subcategory $\mathcal{D} \subseteq$ Mod-R *has enough pure projectives* [19] if for every module $D_R \in \mathcal{D}$, there exists a pure epimorphism $g : P \to D$, where P is a pure projective module in $\mathcal{D}$.

Lemma 1.8. *If* $0 \to H \to M \to B \to 0$ *is an exact sequence with* H *finitely generated and* M *pure projective, then* B *is pure projective.*

Proof. First recall that if M is finitely presented and H is finitely generated, then B is also finitely presented. For the general case, take M' so that $M \oplus M' =$

$\bigoplus_i M_i$ is a direct sum of finitely presented modules. Because H is finitely generated, it is a submodule of a finite sum of the M_i, and so we can assume it is a submodule of just one of them, say M_0. Then $M/H \oplus M' = (M \oplus M')/H = M_0/H \oplus \bigoplus_{i \neq 0} M_i$ is a direct sum of finitely presented modules. □

Theorem 1.9. *The following are equivalent for a torsion pair* $(\mathcal{T}, \mathcal{F})$ *in* Mod-R *with* $\mathcal{T}$ *definable:*

1. *the module* R_R *has a* $\mathcal{T}$*-preenvelope that is pure projective;*
2. *every finitely presented module has a pure projective* $\mathcal{T}$*-preenvelope;*
3. $\mathcal{T}$ *has enough pure projectives;*
4. $\mathcal{T} = \operatorname{Gen} P$ *for some pure projective module.*

Moreover, if these conditions are satisfied, then $(\mathcal{T}, \mathcal{F})$ *is a coherent torsion pair.*

Proof. (1) ⇒ (2). Let $\varepsilon : R \to T_R$ be a pure projective $\mathcal{T}$-preenvelope. We will use the lemma to build a pure projective $\mathcal{T}$-preenvelope of a finitely presented module A. There is a short exact sequence $0 \to H \to R^n \xrightarrow{\pi} A \to 0$ with H a finitely generated module. Take the pushout of π and the pure projective $\mathcal{T}$-preenvelope $\varepsilon^n : R^n \to T_R^n$ to obtain the commutative diagram

(1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & R^n & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varepsilon^n} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & H & \longrightarrow & T_R^n & \longrightarrow & T_A & \longrightarrow & 0. \end{array}$$

The morphism $\delta : A \to T_A$ is then a $\mathcal{T}$-preenvelope of A, with T_A pure projective, by Lemma 1.8.
(2) ⇔ (3). This is just [19, Theorem 8].
(2) ⇒ (4). Let $\mathcal{S}$ be a set of representatives of the isoclasses of finitely presented modules and P the direct sum of the pure projective $\mathcal{D}$-preenvelopes. Then $\mathcal{T} \subseteq \operatorname{Gen} P$. The converse inclusion is clear, since $P \in \mathcal{T}$ and $\mathcal{T}$ is a torsion class.
(4) ⇒ (1). Rada and Saorín [32, Theorem 3.3] proved that every subcategory closed under direct products and pure submodules is preenveloping. Every definable subcategory is therefore preenveloping. Let $\varepsilon : R \to T_R$ be a $\mathcal{T}$-preenvelope of R. By hypothesis, there is an epimorphism $\phi : P^{(I)} \to T_R$ from a coproduct of copies of P. The $\mathcal{T}$-preenvelope ε then factors through ϕ, and yields the module $P^{(I)}$ as a pure projective $\mathcal{T}$-preenvelope of R_R.
To prove the last statement, use Condition (4) to see that $\mathcal{F} = \{M \in \text{Mod-}R \mid \operatorname{Hom}_R(P, M) = 0\}$. As P is pure projective, $\mathcal{F}$ is closed under pure epimorphic images. By Proposition 1.2(3), it is definable. □

Conjecture 1.10. *If* $(\mathcal{T}, \mathcal{F})$ *is a coherent torsion pair, then* $\mathcal{T}$ *has enough pure projective modules.*

If R is right noetherian, the conjecture can be verified by expressing a torsion module $T = \varinjlim M_i$ as the direct limit of its finitely generated submodules.

Then $T = \varinjlim t(M_i)$ is the direct limit of its finitely generated torsion submodules. These are all finitely presented and T is a pure epimorphic image of the pure projective module obtained by taking the direct sum of the $t(M_i)$.

2. Pure projective tilting modules

Definition 2.1. A right R-module T is a (large) 1-*tilting* module if it satisfies the following conditions:

(T1) $\mathrm{pd}(T) \leq 1$;
(T2) $\mathrm{Ext}^1_R(T, T^{(\lambda)}) = 0$, for every cardinal λ;
(T3) there exists an exact sequence:

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0,$$

where T_0, $T_1 \in \mathrm{Add}\, T$. The notion of a 1-*cotilting* module is defined dually.

By [7] a module T is 1-tilting if and only if $\mathcal{T} = T^{\perp} = \mathrm{Gen}\, T$, which is called the tilting class of T. By [14, Lemma 14.2], $\mathcal{T}$ is a torsion class in Mod-R and gives rise to the *tilting* torsion pair $(\mathcal{T}, \mathcal{F})$, where

$$\mathcal{F} = \{M \in \text{Mod-}R \mid \mathrm{Hom}_R(T, M) = 0\}.$$

Two 1-tilting modules T and U are said to be *tilting equivalent* if $T^{\perp} = U^{\perp}$ or, equivalently, if $\mathrm{Add}\, T = \mathrm{Add}\, U$. A 1-tilting module equivalent to a finitely presented 1-tilting module is called *classical.* For later reference, we recall a useful criterion for a module U to be a 1-tilting module equivalent to a given one T.

Lemma 2.2. *Let T be a 1-tilting module and suppose there is a short exact sequence*

$$0 \longrightarrow R \longrightarrow U_0 \longrightarrow U_1 \longrightarrow 0,$$

where U_0, $U_1 \in \mathrm{Add}\, T$. Then $U = U_0 \oplus U_1$ is a 1-tilting module equivalent to T.

Proof. (cf. the proof of [14, Theorem 13.18]). Every module in $\mathrm{Add}\, T$ satisfies Conditions (T1) and (T2) for a 1-tilting module. This is because the class of modules that satisfy Condition (T1) is closed under direct sums and summands. Similarly, the class of modules that satisfy Condition (T2) is clearly closed under direct summands, and it is readily verified that for every index set I, the direct sum $T^{(I)}$ of copies of T also satisfies Condition (T2). Because U belongs to $\mathrm{Add}\, T$ it satisfies the Conditions (T1) and (T2). It satisfies Condition (T3) by hypothesis, so that U is itself a 1-tilting module. Therefore $\mathrm{Gen}\, U = U^{\perp}$. Now $U \in \mathrm{Add}\, T$ implies that the tilting class of U, $\mathrm{Gen}\, U \subseteq \mathrm{Gen}\, T$ is contained in that of T. On the other hand, U is a direct summand of some direct sum $T^{(I)}$, so that $U^{\perp} \supseteq (T^{(I)})^{\perp} = T^{\perp}$, as required. □

If $I \subseteq R$ is a two sided ideal, then we may think of the module category Mod-R/I as a full subcategory of Mod-R induced by restriction of scalars along the quotient map $R \to R/I$ of rings. It consists of the modules $M \in \text{Mod-}R$ for which $MI = 0$. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in Mod-R, with torsion ideal $I = t(R) \subseteq R$, we have that for every $F \in \mathcal{F}$, $\text{Hom}_R(I, F) = 0$, hence $\text{Hom}_R(R/I, F) \cong F$, that is, $FI = 0$. We may therefore think of $\mathcal{F}$ as a subcategory of Mod-R/I.

Proposition 2.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* Mod-R, *with torsion radical t and torsion ideal $I = t(R)$. The torsion free class $\mathcal{F}$ is closed under direct limits in* Mod-R *if and only if there exists a* 1*-cotilting R/I-module C, such that*

$$\mathcal{F} = \{M \in \text{Mod-}R \mid MI = 0 \text{ and } \text{Ext}^1_{R/I}(M, C) = 0\}.$$

Proof. As a consequence of the result that every 1-cotilting module C_R is pure injective [2], the cotilting class ${}^\perp C \subseteq \text{Mod-}R/I$ is a definable subcategory. It follows that $\mathcal{F} \subseteq \text{Mod-}R$ is definable.
For the converse, note that submodules, direct products and direct limits of modules in $\mathcal{F}$ are R/I-modules. Thus, we may consider $\mathcal{F}$ as a definable subcategory of Mod-R/I, which is therefore closed under pure epimorphic images in Mod-R/I. Moreover, $\mathcal{F}$ contains all the projective R/I-modules, thus by [10, Proposition 5.2.2] or [20, Theorem 2.5], $\mathcal{F}$ is a covering class in Mod-R/I. Every cover is an epimorphism, since $\mathcal{F}$ contains the projective R/I-modules. As $\mathcal{F}$ is closed under extensions, Wakamatsu's Lemma implies that $\mathcal{F}$ is a special precovering class in Mod-R/I. By [14, Theorem 15.22], there is a 1-cotilting R/I-module C such that $\mathcal{F} = \text{Ker}\,\text{Ext}^1_{R/I}(-, C)$. □

Note that Proposition 2.3 also follows from [27, Proposition 5.7]. Indeed, $\mathcal{F}$ is closed under direct limits in Mod-R if and only if it is closed under direct limits in Mod-R/I and in this case $\mathcal{F}$ generates Mod-R/I.
In a tilting torsion pair $(\mathcal{T}, \mathcal{F})$, generated by a 1-tilting module T, the tilting class $\mathcal{T} = \text{Gen}\,T$ is definable. This was proved by the first author and Herbera [3], by giving a collection $\mathcal{A} \subseteq \text{mod-}R$ of finitely presented modules of projective dimension at most 1, such that $\mathcal{T} = \mathcal{A}^\perp$. If A_R is a finitely presented module of projective dimension at most 1, then the functor $\text{Ext}^1_R(A, -)$ is coherent, so that $\mathcal{T}$ is definable by Condition (2) of Proposition 1.2.

Theorem 2.4. *The following are equivalent for a* 1*-tilting module T, with tilting class $\mathcal{T} = \text{Gen}\,T$:*

(1) *the module T is pure projective;*
(2) *the definable subcategory $\mathcal{T}$ has enough pure projective modules;*
(3) *the tilting torsion pair $(\mathcal{T}, \mathcal{F})$ is coherent;*
(4) $\mathcal{T} \cap {}^\perp\mathcal{T} \subseteq \text{Add}(\text{mod-}R)$;
(5) *the module R_R admits a pure projective special $\mathcal{T}$-preenvelope;*
(6) *every finitely presented module admits a special pure projective $\mathcal{T}$-preenvelope;*
(7) *every finitely presented module in ${}^\perp\mathcal{T}$ admits a special pure projective $\mathcal{T}$-preenvelope in* Add T; *and*

(8) *T is tilting equivalent to a countably presented pure projective 1-tilting module.*

If R is a right noetherian ring, these conditions are equivalent to the condition that $\mathcal{X} = \mathcal{T} \cap \text{mod-}R$ is a covariantly finite subcategory of mod-R and $\mathcal{T} = \varinjlim(\mathcal{X})$.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1). The first two implications follow from Theorem 1.9; the third from Corollary 1.7; and the fourth from the general fact about a 1-tilting module T that $\mathcal{T} \cap {}^{\perp}\mathcal{T} = \operatorname{Add} T$.

(1) ⇒ (5). By Condition (T3) for a 1-tilting module, there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where both T_0 and T_1 are in $\operatorname{Add} T = \mathcal{T} \cap {}^{\perp}\mathcal{T}$. Thus $0 \to R \to T_0$ is a special $\mathcal{T}$-preenvelope of R with T_0 pure projective.

(5) ⇒ (6). Let A be a finitely presented module. Consider a short exact sequence $0 \to H \to R^n \xrightarrow{\pi} A \to 0$ with H a finitely generated module, and argue as in the proof of (1) ⇒ (2) of Theorem 1.9, by taking the pushout (1) of π and ε^n, where $\varepsilon : R \to T_0$ is a special pure projective $\mathcal{T}$-preenvelope. By the properties of a pushout, the morphisms $\varepsilon^n : R^n \to T_0^n$ and $\delta : A \to T_A$ are also special pure projective $\mathcal{T}$-preenvelopes.

(6) ⇒ (7). If A is a finitely presented module in ${}^{\perp}\mathcal{T}$, then the special $\mathcal{T}$-preenvelope given by the hypothesis lies in $\mathcal{T} \cap {}^{\perp}\mathcal{T} = \operatorname{Add} T$.

(7) ⇒ (2). Apply the hypothesis to the module R_R and apply the implication (1) ⇒ (3) of Theorem 1.9.

(1) ⇔ (8). By assumption, every module T_0 in $\operatorname{Add} T$ is pure projective, a direct summand of a direct sum $\oplus_{i \in I} E_i$ with E_i finitely presented modules, hence in particular countably generated. By Kaplansky's Theorem [23, Theorem 1], T_0 is a direct sum of countably generated submodules X_α in $\operatorname{Add} T$, $T_0 = \oplus_{\alpha \in \Lambda} X_\alpha$. By Condition (T3) for a 1-tilting module, there is a short exact sequence $0 \to R \xrightarrow{\varepsilon} T_0 \to T_1 \to 0$ with T_0 as described. The image of ε is contained in a summand U_0 of T, $U_0 = \oplus_{\beta \in F_0} X_\beta$ where F_0 is a finite subset of Λ. Hence, U_0 is countably generated and $T_1 \cong U_0/\varepsilon(R) \oplus \oplus_{\alpha \in \Lambda \setminus F_0} X_\alpha$. The quotient module $U_1 = U_0/\varepsilon(R)$ is a summand of T_1 and so also belongs to $\operatorname{Add} T$. It follows from Lemma 2.2 that $U = U_0 \oplus U_1$ is a countably generated 1-tilting module in $\operatorname{Add} T$ that is tilting equivalent to T. By [4, Corollary 3.9], a countably generated 1-tilting module is countably presented, that is, admits a presentation by countably generated projective modules.

If the ring R is right noetherian, then the condition in the last statement follows from the remark in the last paragraph of Section 1, together with [14, Lemma 8.35]; the converse from [14, Lemma 8.36]. □

From a tilting torsion pair $(\mathcal{T}, \mathcal{F})$ in a Grothendieck category $\mathcal{G}$ there arises [17, §I.2] a corresponding torsion pair $(\mathcal{F}[1], \mathcal{T}[0])$ in the heart $\mathcal{H}_t \subseteq \mathcal{D}^b(\mathcal{G})$ of the HRS t-structure with the property that $\mathcal{F}[1] \cong \mathcal{F}$ is equivalent to the given torsion free class, while $\mathcal{T}[0] \cong \mathcal{T}$ is equivalent to the torsion class. If $\mathcal{G} = \text{Mod-}R$ is a module category and $\mathcal{T} = \operatorname{Gen} T$ for some 1-tilting module T, then the stalk complex $T[0]$ is a projective generator of $\mathcal{H}_t$.

Corollary 2.5. *Let T be a 1-tilting module. The heart $\mathcal{H}_t$ of the HRS t-structure in the bounded derived category $\mathcal{D}^b(\text{Mod-}R)$ induced by the torsion pair $(\operatorname{Gen} T, \mathcal{F})$ is a Grothendieck category if and only if T is a pure projective module.*

Proof. Since T is a 1-tilting module, the torsion class $\operatorname{Gen} T$ is definable, by [3]. By [28, Theorem 1.2], the heart $\mathcal{H}_t$ is a Grothendieck category if and only if $\mathcal{F}$ is definable, and this is equivalent to T being pure projective, by the equivalence (1) $\Leftrightarrow$ (3) of Theorem 2.4. □

The following is the *classical* version of Theorem 2.4.

Theorem 2.6. *Let T be 1-tilting module with tilting class $\mathcal{T} = \operatorname{Gen} T$. The following are equivalent:*

1. *T is classical;*
2. *$\mathcal{T} \cap {}^{\perp}\mathcal{T} \subseteq \operatorname{Add}(\mathcal{T} \cap {}^{\perp}\mathcal{T} \cap \text{mod-}R)$;*
3. *the module R_R admits a finitely presented special $\mathcal{T}$-preenvelope;*
4. *every finitely presented module admits a finitely presented special $\mathcal{T}$-preenvelope; and*
5. *every finitely presented module in ${}^{\perp}\mathcal{T}$ admits a finitely presented special $\mathcal{T}$-preenvelope in $\operatorname{Add} T$.*

In case these conditions hold, the subcategory $\mathcal{X} = \mathcal{T} \cap \text{mod-}R$ is covariantly finite in mod-R and $\mathcal{T} = \varinjlim(\mathcal{X})$.

Proof. (1) $\Rightarrow$ (2). If T is equivalent to a finitely presented module E, then

$$\mathcal{T} \cap {}^{\perp}\mathcal{T} = \operatorname{Add} T = \operatorname{Add} E \subseteq \operatorname{Add}(\mathcal{T} \cap {}^{\perp}\mathcal{T} \cap \text{mod-}R).$$

(2) $\Rightarrow$ (3). Condition (T3) implies that there is a short exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with T_0 and T_1 in $\operatorname{Add} T$. By hypothesis, T_0 is a direct summand of a direct sum $T_0 \oplus T_0' = \oplus_{i\in I} E_i$ with E_i finitely presented modules in $\operatorname{Add} T$. Adding to both of the modules T_0 and T_1 the direct summand T_0', we may assume that there exists a sequence $0 \to R \to V_0 \to V_1 \to 0$ with V_0 and V_1 in $\operatorname{Add} T$ and V_0 is a direct sum $\oplus_{i\in I} E_i$ of finitely presented modules E_i. Arguing as in the proof of (1) $\Rightarrow$ (8) of Theorem 2.4 yields a short exact sequence $0 \to R \to U_0 \to U_1 \to 0$ where U_0 and U_1 are finitely presented modules in $\operatorname{Add} T = \mathcal{T} \cap {}^{\perp}\mathcal{T}$. Thus $0 \to R \to U_0$ is a special $\mathcal{T}$-preenvelope of R with U_0 finitely presented.
(3) $\Rightarrow$ (4). Let A be a finitely presented module and argue as in the proof of (1) $\Rightarrow$ (2) of Theorem 2.4, with $\varepsilon : R \to T_0$ a special finitely presented $\mathcal{T}$-preenvelope. Then ε^n is also such and $\delta : A \to T_A$ is the required special finitely presented $\mathcal{T}$-preenvelope of A.
(4) $\Rightarrow$ (5). Clear, since a special $\mathcal{T}$-preenvelope of a module in ${}^{\perp}\mathcal{T}$ belongs to $\operatorname{Add} T$.
(5) $\Rightarrow$ (1). Let $0 \to R \to U_0 \to U_1 \to 0$ be a special $\mathcal{T}$-preenvelope of R with U_0 finitely presented. Then both U_0 and U_1 belong to $\mathcal{T} \cap {}^{\perp}\mathcal{T} = \operatorname{Add} T$, so that $U = U_0 \oplus U_1$ is, by Lemma 2.2, a 1-tilting module equivalent to T.

For the last statement, note that Condition (4) implies that $\mathcal{T} \cap \text{mod-}R$ is covariantly finite and use Lenzing's special case [24] of [19, Theorem 8]. □

Recall from [7] that a module X is *self small* if the canonical morphism $\mathrm{Hom}_R(X,X)^{(I)} \to \mathrm{Hom}_R(X,X^{(I)})$ is an isomorphism, for every index set I. If T is a pure projective 1-tilting module, then, because the tilting class $\mathrm{Gen}\, T$ is cogenerating, Proposition 5.3 of [27] implies that the heart $\mathcal{H}_t \subseteq \mathcal{D}^b(\text{Mod-}R)$ of the HRS t-structure associated to the torsion pair $(\mathrm{Gen}\, T, \mathcal{F})$ is a module category if and only if the tilting class $\mathrm{Gen}\, T$ may be represented by a self-small tilting module T. By [7, Proposition 1.3] a 1-tilting module is self-small if and only if it is finitely presented.

Corollary 2.7. *Let T be a 1-tilting module. The heart $\mathcal{H}_t$ of the HRS t-structure in $\mathcal{D}^b(\text{Mod-}R)$ induced by the torsion pair $(\mathrm{Gen}\, T, \mathcal{F})$ is equivalent to a module category if and only if T is classical.*

Corollary 2.8. *If R is a ring over which every right pure projective module is a direct sum of finitely presented modules, then every pure projective 1-tilting module is classical.*

Proof. If T is a pure projective 1-tilting module over such a ring, with tilting class $\mathcal{T}$, then $\mathcal{T} \cap {}^{\perp}\mathcal{T} = \mathrm{Add}\, T$ consists of pure projective modules, so that the hypothesis implies Condition (2) of Theorem 2.6. □

If $A \in \text{mod-}R$ is a finitely presented module of projective dimension at most 1, then the functor $\mathrm{Ext}^1_R(A,-) : \text{mod-}R \to \mathrm{Ab}$ is coherent so that the condition for a module M_R to satisfy $\mathrm{Ext}^1_R(A,M) = 0$ is elementary [31, §10.2.6]: there is a sentence σ_A in $\mathcal{L}(R)$ such that $\mathrm{Ext}^1_R(A,M) = 0$ if and only if $M \models \sigma_A$.

Proposition 2.9. *Let T be a 1-tilting module. The tilting class $\mathcal{T} = \mathrm{Gen}\, T = T^{\perp}$ is finitely axiomatizable if and only if there exists a finitely presented module A_R of projective dimension at most 1 such that $\mathcal{T} = A^{\perp}$.*

Proof. If $\mathcal{T} = A^{\perp}$, then $\mathcal{T}$ is axiomatized by the sentence σ_A. For the converse, suppose that $\mathcal{T}$ is finitely axiomatized by the sentence σ, and let $\mathcal{A} \subseteq \text{mod-}R$ be an additive subcategory of modules of projective dimension at most 1 for which $\mathcal{T} = \mathcal{A}^{\perp}$. The collection $\mathrm{Th}(\text{Mod-}R) \cup \{\neg\sigma\} \cup \{\sigma_A \mid A \in \mathcal{A}\}$ of sentences in $\mathcal{L}(R)$ is inconsistent. By the Compactness Theorem, some finite subcollection is already inconsistent, which implies that there are finitely many $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $\mathrm{Th}(\text{Mod-}R) \vdash \bigwedge_{i=1}^n \sigma_{A_i} \to \sigma$. In other words, if $A = \oplus_{i=1}^n A_i$, then $\mathrm{Ext}^1_R(A,M) = 0$ implies $M \in \mathcal{T}$. □

According to Theorem 2.4(8), a pure projective 1-tilting module is equivalent to a countably presented pure projective 1-tilting module T. For many purposes, we may thus replace the given 1-tilting module by T and extract some further information when T is pure projective. Arguing as in [37, Lemma 4.4], every countably presented 1-tilting module T, with tilting class $\mathcal{T}$ may be represented as the limit of a linear system $T = \lim\limits_{n\to\infty} A_n$, where for each n :

(1) $A_n \in \text{mod-}R$;
(2) $\text{pd}(A_n) \leq 1$; and
(3) $A_n \in {}^{\perp}\mathcal{T}$.

Such a linear system $(A_n, f_n : A_n \to A_{n+1})_{n\in\mathbb{N}}$ is called a *system associated* to T; it yields a pure exact sequence

$$0 \longrightarrow \oplus_{n\in\mathbb{N}} A_n \xrightarrow{\phi} \oplus_{n\in\mathbb{N}} A_n \longrightarrow T \longrightarrow 0$$

in the usual way, where $\phi(a_1, a_2, \cdots, a_n, \cdots) = (a_1, a_2 - f_1(a_1), \cdots, a_n - f_{n-1}(a_{n-1}), \cdots)$.

Proposition 2.10. *If T is a countably presented pure projective 1-tilting module with associated system $(A_n | n \in \mathbb{N})$, then*

(1) $T \oplus (\oplus_{n\in\mathbb{N}} A_n) \cong \oplus_{n\in\mathbb{N}} A_n$;
(2) *there exists a $k \in \mathbb{N}$ such that $B_k = \oplus_{n\leq k} A_n$ satisfies $B_k^{\perp} = \mathcal{T}$; and*
(3) $\lim\limits_{n\to\infty} A_n/t(A_n) = 0$.

Proof. Because T is pure projective, the pure exact sequence above splits and yields (1). This implies that $\bigcap_n A_n^{\perp} \subseteq T^{\perp} \subseteq A_k^{\perp}$ for every $k \in \mathbb{N}$, since $A_k \in {}^{\perp}\mathcal{T}$. The first inclusion is thus an equality. As T is pure projective, $\mathcal{T} = T^{\perp}$ is finitely axiomatizable, so that we can argue as in the proof of Proposition 2.9 to get (2). Finally, the pure projective assumption on T also entails, by Theorem 1.4, that the torsion radical $t : \text{mod-}R \to \text{mod-}R$ is coherent, and so respects direct limits. This implies that $\lim\limits_{n\to\infty} t(A_n) = t(\lim\limits_{n\to\infty} A_n) = t(T) = T$. Consider the linear system of short exact sequences

$$0 \longrightarrow t(A_n) \longrightarrow A_n \longrightarrow A_n/t(A_n) \longrightarrow 0$$

associated to the system $(A_n | n \in \mathbb{N})$ and take the limit to obtain the short exact sequence

$$0 \longrightarrow T \xrightarrow{1_T} T \longrightarrow \lim_{n\to\infty} A_n/t(A_n) \longrightarrow 0,$$

which establishes (3). □

3. The Commutative case

In this section we prove that a pure projective 1-tilting module over a commutative ring is projective. The result involves the notion of the henselization of local rings, but for some classes of commutative rings, like noetherian or arithmetic rings the arguments are simpler. First, recall that a 1-tilting module T over a commutative ring is equivalent to a classical tilting module if and only if T is projective (see for instance [29, Lemma 1.2]).

Lemma 3.1. *Let T be a 1-tilting module over a commutative ring R and let S be a multiplicative subset of R. Then T_S is a 1-tilting R_S-module and if T is pure projective, so is T_S.*

Proof. By [14, Proposition 13.50], T_S is a 1-tilting R_S-module. (Note that the proof given there becomes simpler in the case of 1-tilting modules, since the first syzygy of T is projective). If T is pure projective, it follows immediately that T_S is pure projective, since R_S is a flat R-module. □

The next proposition allows to reduce the investigation to the case of local commutative rings.

PROPOSITION 3.2. *Let T be a 1-tilting module over a commutative ring R. Then T is projective if and only if $T_\mathfrak{m}$ is a projective 1-tilting $R_\mathfrak{m}$-module, for every $\mathfrak{m} \in \operatorname{Max} R$.*

Proof. By Lemma 3.1, $T_\mathfrak{m}$ is a 1-tilting $R_\mathfrak{m}$-module, for every $\mathfrak{m} \in \operatorname{Max} R$. If T is projective, then clearly $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module.
For the converse, let us apply again the result [3] that every 1-tilting module T is of finite type, that is, there exists a set $\{A_i; i \in I\}$ of finitely presented modules with $\operatorname{pd}(A_i) \leq 1$ such that $M \in T^\perp$ if and only if $\operatorname{Ext}^1_R(A_i, M) = 0$, for every $i \in I$. For every maximal ideal $\mathfrak{m}$ of R we have $\operatorname{Ext}^1_{R_\mathfrak{m}}((A_i)_\mathfrak{m}, T_\mathfrak{m}) = 0$. This implies that $(A_i)_\mathfrak{m} \in {}^\perp(T_\mathfrak{m}^\perp)$. By assumption, $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module. Thus $(A_i)_\mathfrak{m}$ is projective, too. We conclude that A_i is a projective R-module, for every $i \in I$. Hence T is projective. □

A module M over a ring R is FP_2 if M is finitely presented and a first syzygy of M is finitely presented.

LEMMA 3.3. *Let $(R, \mathfrak{m})$ be a commutative local ring and let A be an FP_2-module R-module. The following are equivalent:*

(1) *A is projective;*
(2) $\operatorname{Tor}^R_1(A, R/\mathfrak{m}) = 0$;
(3) $\operatorname{Ext}^1_R(A, R/\mathfrak{m}) = 0$.

Proof. (1) ⇔ (2). This is well known (see for instance [13, Lemma 2.5.8]).
(2) ⇔ (3). For every R-module M let $M^* = \operatorname{Hom}_R(M, E)$ where E is an injective envelope of $R/\mathfrak{m}$. Then $(R/\mathfrak{m})^* \cong R/\mathfrak{m}$ and by well known homological formulas, $\operatorname{Ext}^1_R(A, R/\mathfrak{m}) = 0$ if and only if $\operatorname{Tor}^R_1(A, R/\mathfrak{m}) = 0$. □

PROPOSITION 3.4. *Let $(R, \mathfrak{m})$ be a local commutative ring. A 1-tilting module T is projective if and only if $R/\mathfrak{m} \in T^\perp$. So if T is not projective, then no nonzero finitely generated module is torsion.*

Proof. Necessity is clear. As in the proof of Proposition 3.2, let $\{A_i, i \in I\}$ be a set of finitely presented modules of projective dimension at most one such that $T^\perp = (\bigoplus_{i \in I} A_i)^\perp$. By assumption $\operatorname{Ext}^1_R(A_i, R/\mathfrak{m}) = 0$ for every $i \in I$. By Lemma 3.3 we conclude that every A_i is projective, hence $T^\perp = \text{Mod-}R$ and T is projective.
If T is not projective and $M \in \mathcal{T}$ is a nonzero finitely generated torsion module, Nakayama's Lemma implies that the nonzero quotient $M/M\mathfrak{m}$ and therefore $R/\mathfrak{m}$ belongs to $\mathcal{T}$. □

We can use Proposition 3.4 to see that every pure projective 1-tilting module T over a commutative noetherian ring is projective. By Proposition 3.2 and Theorem 2.4, it suffices to verify the case when R is a local and T countably presented. If $(A_n | n \in \mathbb{N})$ is a system associated to T, then Proposition 2.10(1) implies that $T \oplus (\oplus_{n\in\mathbb{N}} A_n) = \oplus_{n\in\mathbb{N}} A_n$. Because $t(T) = T$, the submodule $t(A_n) \subseteq A_n$ must be nonzero for some n. As R is noetherian, this implies that $t(A_n)$ is a finitely generated torsion module. By Proposition 3.4, T must be projective.

Proposition 3.5. *If R is a commutative local Krull-Schmidt ring, then every pure projective* 1*-tilting module over R is projective.*

Proof. Over a Krull-Schmidt ring, every finitely presented module is a direct sum of indecomposable modules with a local endomorphism ring, so that every pure projective module T is a direct sum of finitely presented modules. If T were not projective, then by Proposition 3.4, none of these finitely presented modules would be in $\mathcal{T}$, which is absurd. □

Recall that a commutative ring R is a *chain ring* if the lattice of its ideals is linearly ordered and R is *arithmetic* if the lattice of its ideals is distributive. By [21, Theorem 1], a ring R is arithmetic if and only every localization of R at a maximal ideal is a chain ring. It is well known that if R is a chain ring, then every finitely presented module is a direct sum of cyclically presented modules (see e.g. [22, Theorem 9.1]), that is, modules of the form R/rR, for some $r \in R$. The endomorphism rings of such modules are clearly local, so that every chain ring is Krull-Schmidt. Propositions 3.2 and 3.5 imply that every pure projective 1-tilting module over an arithmetic ring is projective.
Recall that a local commutative ring $(R, \mathfrak{m})$ with residue field k is *henselian* if for every monic polynomial $f \in R[X]$ and every factorization $\overline{f} = g_0 h_0$ in $k[X]$ with g_0 and h_0 comaximal (i.e., they generate $k[x]$), there is a factorisation $f = gh$ in $R[X]$ such that $\overline{g} = g_0$ and $\overline{h} = h_0$. Examples of henselian rings include 0-dimensional local rings and local rings $(R, \mathfrak{m})$ which are complete in the $\mathfrak{m}$-adic topology. An important result about henselian rings is that they are Krull-Schmidt (see [34] or [11, V Section7]).
In order to prove the main result of this section, we need to recall that every commutative local ring admits a *henselization* (see [26, Chapter VII] or [15, 18.6]).

Proposition 3.6. *Let $(R, \mathfrak{m})$ be a local commutative ring. There is a local ring R^H and a local ring homomorphism $h : R \to R^H$, such that:*

(1) *R^H is henselian;*
(2) *$R \to R^H$ is faithfully flat;*
(3) *$\mathfrak{m}R^H$ is the maximal ideal of R^H;*
(4) *for every ring homomorphism $f : R \to R'$ with R' henselian, there is a unique ring homomorphism $g : R^H \to R'$ such that $f = g \circ h$.*

Theorem 3.7. *Let R be a commutative ring. Then every pure projective* 1*-tilting module is projective, and therefore classical.*

Proof. By Proposition 3.2 we can assume that R is local. Let T be a 1-tilting R-module and consider a henselization R^H of R. Since R^H is flat, we can argue as in the proof of [14, Proposition 13.50], to conclude that the R^H-module $T^H = T \otimes_R R^H$ is 1-tilting. Moreover, again by flatness, if T is pure projective so is T^H. Since henselian rings are Krull-Schmidt, Proposition 3.5 gives that T^H is a projective R^H-module and since projectivity descends along faithfully flat ring homomorphisms (see [33, Part II], corrected in [16], or by [35, 10.92.1]), we conclude that T is a projective R-module. □

4. The Noetherian case

In this section we show how to construct examples of nonclassical pure projective 1-tilting modules over noetherian rings. The construction is based on the following result of J. Whitehead [38].

Theorem 4.1. *Let I be an idempotent ideal of R finitely generated on the left. Then there exists a countably generated projective module $P \in$ Mod-R with trace ideal I.*

Let us briefly explain how to construct P. Suppose that $I = Ri_1 + \cdots + Ri_k$ and let $c = (i_1, \ldots, i_k)^T$ be a column containing the generators of I. For every $n \in \mathbb{N}$ let $F_n = R^{k^{n-1}}$ and let $f_n \colon F_n \to F_{n+1}$ be the homomorphism given by the block-diagonal matrix having every diagonal block equal to c. For example, if $k = 2$ we have

$$f_1 = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \times -, f_2 = \begin{pmatrix} i_1 & 0 \\ i_2 & 0 \\ 0 & i_1 \\ 0 & i_2 \end{pmatrix} \times -$$

Then it is possible to show the existence of homomorphisms $g_n \colon F_{n+1} \to F_n$ such that $f_n = g_{n+1} f_{n+1} f_n$ for every $n \in \mathbb{N}$. Let $P = \varinjlim F_i$, then the canonical presentation of the direct limit in the short exact sequence

$$0 \longrightarrow \oplus_{i\in\mathbb{N}} F_i \longrightarrow \oplus_{i\in\mathbb{N}} F_i \longrightarrow P \longrightarrow 0$$

splits, so P is projective.

Our aim is to modify the construction a bit to obtain a pure projective 1-tilting module. From now on assume that $I = Ri_1 + \cdots + Ri_k$ is an idempotent ideal satisfying the following property: If $Ir = 0$ for some $r \in R$ then $r = 0$. Observe that this property holds if and only if $\varepsilon_{l+1} = f_l \cdots f_2 f_1$ is a monomorphism for every $l \in \mathbb{N}$. Further let $\pi_l \colon F_l \to M_l$ be the cokernel of ε_l. Consider the following diagram whose columns are short exact sequences

$$\begin{array}{ccccccccc}
F_1 & = & F_1 & = & F_1 & = & F_1 & = & \cdots \\
\| & & \downarrow{\scriptstyle \varepsilon_2} & & \downarrow{\scriptstyle \varepsilon_3} & & \downarrow{\scriptstyle \varepsilon_4} & & \\
F_1 & \xrightarrow{f_1} & F_2 & \xrightarrow{f_2} & F_3 & \xrightarrow{f_3} & F_4 & \xrightarrow{f_4} & \cdots \\
\downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_4} & & \\
0 & \longrightarrow & M_2 & \xrightarrow{\overline{f_2}} & M_3 & \xrightarrow{\overline{f_3}} & M_4 & \xrightarrow{\overline{f_4}} & \cdots
\end{array}$$

Considering the direct limits of the rows in this diagram we obtain an exact sequence

$$0 \longrightarrow R \xrightarrow{\varepsilon} P \xrightarrow{\pi} Q \longrightarrow 0,$$

where P is projective and Q is pure projective by Lemma 1.8. In fact, it is easy to see that every homomorphism $g_n \colon F_{n+1} \to F_n$ induces a homomorphism $\overline{g_n} \colon M_{n+1} \to M_n$ such that $\overline{g_{n+1} f_{n+1}}\ \overline{f_n} = \overline{f_n}$ holds for every $n \geq 2$. Then the canonical presentation of Q as $\varinjlim M_n$ splits, in particular Q is a direct summand of $\oplus_{2 \leq i \in \mathbb{N}} M_i$.

Proposition 4.2. *The module $T = P \oplus Q$ constructed above is pure projective 1-tilting and such that*

$$T^{\perp} = \operatorname{Gen} T = \{M \in \text{Mod-}R \mid MI = M\}.$$

Proof. It is easy to see and well known that $\operatorname{Gen} T = \operatorname{Gen} P = \{M \in \text{Mod-}R \mid MI = M\}$ since I is the trace ideal of P. Further observe that $T^{\perp} = Q^{\perp}$ and $0 \to R \overset{\varepsilon}{\to} P \overset{\pi}{\to} Q \to 0$ is a projective presentation of Q. Therefore if $N \in Q^{\perp}$, then every homomorphism $f \colon R \to N$ is of the form $f'\varepsilon$ for some $f' \colon P \to N$. Now $NI = N$ is an easy consequence of $PI = P$. So $T^{\perp} \subseteq \{M \in \text{Mod-}R \mid MI = M\}$.
Thus we are left to prove that if $N \in \{M \in \text{Mod-}R \mid MI = M\}$ then $N \in Q^{\perp}$ that is for every $\varphi \colon R \to N$ there exists $\varphi' \colon P \to N$ such that $\varphi = \varphi'\varepsilon$. Recall that P is a direct limit of the sequence

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3 \xrightarrow{f_3} \cdots .$$

For any $i \in \mathbb{N}$ let $\iota_i \colon F_i \to P$ be the colimit injection. Observe that $F_1 = R$ and $\varepsilon = \iota_1$. Further f_1 is a multiplication by the column consisting of generators of ${}_R I$. Then it is easily verified that for a given homomorphism $\varphi \colon F_1 \to N$ there exists $\psi \colon F_2 \to N$ such that $\psi f_1 = \varphi$. For every $i \geq 2$ put $\varphi_i = \psi g_2 g_3 \cdots g_i f_i \colon F_i \to N$. The property $g_{i+1} f_{i+1} f_i = f_i$ shows that $\varphi_{i+1} f_i = \varphi_i$ for every $i \geq 2$. The universal property of direct limits gives the homomorphism $\varphi' \colon P \to N$ such that $\varphi' \iota_i = \varphi_i$ for every $i \geq 2$. In particular, $\varphi' \iota_2 f_1 = \varphi_2 f_1$. The LHS of this equality is just $\varphi'\varepsilon$ and the RHS is $\varphi_2 f_1 = \psi g_2 f_2 f_1 = \psi f_1 = \varphi$. So $\varphi'\varepsilon = \varphi$ and we are done. □

Recall that a finitely generated module $M \in \text{Mod-}R$ is *stably free* if there exist a finitely generated free module $F \in \text{Mod-}R$ such that $M \oplus F$ is free (finitely generated).

Let us present a criterion for producing a pure projective 1-tilting module that is not classical.

Theorem 4.3. *Let R be a ring and let $I \subseteq R$ be an idempotent ideal finitely generated on the left satisfying $Ir = 0 \Rightarrow r = 0$. Further suppose that the following conditions hold:*

1. *every finitely generated projective right R-module is stably free;*
2. *there exists a proper ideal K containing I such that every finite power of R/K is a directly finite module, that is, it is not isomorphic to a proper direct summand of itself;*
3. *there exists a flat homomorphism $\varphi\colon R \to S$ of rings such that S is a nontrivial semisimple artinian ring and $\varphi(I) \neq 0$.*

If $T \in \text{Mod-}R$ is a 1-tilting module such that $T^{\perp} = \{M \in \text{Mod-}R \mid MI = M\}$, then T is not tilting equivalent to a direct sum of finitely presented modules, hence not classical.

Proof. We claim that every finitely presented module M_R of projective dimension at most 1 satisfying $MI = M$ satisfies $M \otimes_R S = 0$: Let

$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

be a projective presentation of such a module, where P_1, P_2 are finitely generated projectives. By (1) we may assume $P_1 \simeq R^m$ and $P_2 \simeq R^n$ for some $m, n \in \mathbb{N}$. Now apply the functor $- \otimes_R R/K$ to this free presentation of M. Using $MK = M$ we get an epimorphism $(R/K)^m$ onto $(R/K)^n$. Since $(R/K)^m$ is directly finite, $m \geq n$ follows. Finally apply $- \otimes_R S$ to the presentation of M to obtain

$$0 \longrightarrow S^m \longrightarrow S^n \longrightarrow M \otimes_R S \longrightarrow 0.$$

Since $m \geq n$ and S is an S-module of finite length then $m = n$ and $M \otimes_R S = 0$. This proves the claim.
We can complete the proof easily. Let T be a 1-tilting module such that $T^{\perp} = \{M \in \text{Mod-}R \mid MI = M\}$ and $\operatorname{Add} T = \operatorname{Add}(\oplus_{i \in I} M_i)$, where every M_i is a finitely presented module. Since T is 1-tilting, every M_i is of projective dimension at most 1 and $M_i I = M_i$. Because T is a direct summand of a sum of the M_i, the claim implies $T \otimes_R S = 0$. On the other hand, there is a projective module P of the trace ideal I, so $PI = P \in \operatorname{Gen} T$ and P is a direct summand of $T^{(\kappa)}$. In particular, $P \otimes_R S = 0$. But it is not possible if $\varphi(I) \neq 0$. □

Observe that if R is noetherian and $0 \neq I \neq R$ then Condition (2) of Theorem 4.3 holds with $I = K$. Moreover, if R is noetherian semiprime then Condition (3) is a consequence of Goldie's theorem [25, Theorem 2.3.6, Proposition 2.1.16(ii)]. So in the noetherian context we have to care only about the existence of a suitable idempotent ideal I and Condition (1).
Now we can give the promised examples. First, let us consider the universal enveloping algebra of $\mathrm{sl}(2, \mathbb{C})$, that is $R = \mathbb{C}\langle h, e, f\rangle/(h = ef - fe, 2e = he -$

$eh, -2f = hf - fh$). This is a noetherian domain, let I be the ideal generated by h, e, f. Obviously $I^2 = I$ and $R/I \simeq \mathbb{C}$. So we can take $K = I$ in order to check (2). The condition (3) is satisfied as the inclusion of R into its (right) quotient division ring is flat. Finally, according to [25, Corollary 12.3.3] every finitely generated projective R-module is stably free.

We give one more example, this noetherian domain is a bit artificial but on the other hand it is semilocal and its projective modules are classified. For details see [18, Example 5.1]. Let R be a semilocal principal ideal domain such that $R/J(R) \simeq M_3(F) \times M_3(F)$, where F is a field (existence of such a ring follows from the work of Fuller and Shutters [12], see for example [18, Examples 3.3]). Let $\pi\colon R \to M_3(F) \times M_3(F)$ be the canonical surjection and let $\iota\colon F \times F \to M_3(F) \times M_3(F)$ be given by $\iota(x, y) = \mathrm{diag}(x, y, y) \times \mathrm{diag}(x, x, y)$. Consider Λ to be the pullback of the diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ F \times F & \xrightarrow{\ \iota\ } & M_3(F)^2 \end{array}$$

Then Λ is a noetherian domain, $\Lambda/J(\Lambda) \simeq F \times F$, every finitely generated projective right Λ-module is free but there exists a countably generated projective module $P \in \text{Mod-}\Lambda$ such that $0 \neq \mathrm{Tr}(P) \neq R$. So we can apply Theorem 4.3 directly.

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